Sz.-Nagy—Brehmer dilations

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1. Introduction

1. 1. The following discussion (sections 1. 1 and 1. 2) is due to B. SZ.-NAGY [2, 3, 4].[1])

Let J denote a totally ordered set of indices α and let $\tilde{J}$ denote the set of all integer valued functions $n \equiv n(\alpha)$, $-\infty < n(\alpha) < \infty$, such that $\tilde{n} \equiv$ set of α with $n(\alpha) \neq 0$, is a finite subset of J.

Write $n=0$ if $n(\alpha)=0$, $n \leqq m$ if $n(\alpha) \leqq m(\alpha)$, for all α. For given $n, m \in \tilde{J}$ define $n-m, n^+$ by: $(n-m)(\alpha) = n(\alpha)-m(\alpha)$, $n^+(\alpha) = \max(n(\alpha), 0)$ for all α. Call n, m *positive-disjoint* if $(n-m)^+ = n$ and $(m-n)^+ = m$ (then necessarily $n \geqq 0$ and $m \geqq 0$).

Suppose H is a given Hilbert space. If T is an operator[2]) on H and i is an integer, define $T(i)$ to be T^i if $i \geqq 0$ and $(T^*)^{-i}$ if $i<0$. If T_α $(\alpha \in J)$ are given operators on H define $T(n)$ as follows: if the indices in $\tilde{n}$, ordered as in J, are denoted as $1, \ldots, r$, let $T(n) = T_1(n(1))\, T_2(n(2)) \ldots T_r(n(r))$, with the convention: $T(0)$ shall mean 1.

Call a family of unitary operators U_α $(\alpha \in J)$ acting on some Hilbert space $K \supset H$ a *unitary dilation of the* T_α $(\alpha \in J)$ if the U_α $(\alpha \in J)$ are commuting and

(1. 1) $$T(n)x = P_H U(n)x \text{ for all } x \in H \text{ and all } n \geqq 0,$$ [3])

and *minimal* if also

(1. 2) $$K \text{ is spanned by the } U(n)x \text{ (all } x \in H, \text{ all } n \in \tilde{J}).$$ [4])

Note: the U_α are required to be commuting but not the T_α.

1. 2. Suppose T_α $(\alpha \in J)$ has a unitary dilation U_α $(\alpha \in J)$. Then clearly the T_α must be contractions[5]). Consider the scalar valued function $B \equiv B(n, x; m, y)$ *defined*

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[1]) The writer acknowledges with pleasure several stimulating conversations with Prof. SZ.-NAGY.

[2]) The scalars may be the real, complex or quaternionic numbers. Inner product will be denoted $(x|y)$. In this paper all operators are bounded, linear. Operators T_α $(\alpha \in J)$ are called commuting if $T_\alpha T_\beta = T_\beta T_\alpha$ for all $\alpha, \beta \in J$.

[3]) P_H will denote the projection (orthogonal) of K onto H.

[4]) If (1. 1) is satisfied by some K, U_α $(\alpha \in J)$ then (1. 1), (1. 2) will be satisfied by K_0 (the subspace of K spanned by the vectors named in (1. 2)) and the restrictions of the U_α $(\alpha \in J)$ to this K_0.

[5]) A contraction T is a linear operator with $\|T\| \leqq 1$.

for $x, y \in H$ *and positive-disjoint* n, m *by*

(1. 3) $$B \equiv (U(n)x \mid U(m)y).$$

As pointed out by Sz.-Nagy, B has the properties:

(1. 4) $$B(n, x; 0, y) = (T(n)x \mid y).$$

(1. 5) $B(n, x; m, y)$ *is linear in* y *and* $B(n, x; m, y) = \overline{B(m, y; n, x)}$.

(1. 6) *(The positivity condition:)*

$$\sum_{i,j=1}^{N}{}' B((n_i - n_j)^+, x_i; (n_j - n_i)^+, x_j) \geqq 0$$

for all $x_1, \ldots, x_N \in H, \quad n_1, \ldots, n_N \geqq 0, \qquad N = 1, 2, \ldots$.[6])

On the other hand, if T_α $(\alpha \in J)$ are given contractions then, as pointed out by Sz.-Nagy every scalar valued B with properties (1. 4)–(1. 6) determines uniquely a minimal unitary dilation (unique to within a unitary isomorphism). The T_α need not be commuting.

We sketch the proof. Let K' denote the linear space of *formal* finite sums $\sum_i (n_i, x_i)$ with all n_i in $\hat{J}$ and x_i in H[7]) (where, if $v = \sum_i (n_i, x_i)$ and $w = \sum_j (m_j, y_j)$ then $vc = \sum_i (n_i, x_i c)$ for scalar c, and $v \pm w$ have the obvious values). Define $(v \| w)$ as

$$\sum_{i,j} B((n_i - m_j)^+, x_i; (m_j - n_i)^+, y_j).$$

Then $(v \| v) \geqq 0$ for all v in K' (because of (1. 6)). Identify v and w (in K') if

$$((v - w) \| (v - w)) = 0.$$

After such identifications K' becomes a Hilbert space (possibly incomplete) with inner product $(v \mid w) = (v \| w)$. Now identify x in H with $(0, x)$ and so imbed H in K'. For each β, determine a linear operator U_β on K' by the relation: $U_\beta(n, x) = (m, x)$ where $m(\beta) = n(\beta) + 1$ and $m(\alpha) = n(\alpha)$ if $\alpha \neq \beta$. Then the U_α are unitary on K' and their extension to K, the completion of K', is the desired minimal unitary dilation.[8])

1. 3. Thus for given contractions T_α $(\alpha \in J)$ there are precisely as many solutions K, U_α $(\alpha \in J)$ of (1. 1), (1. 2) as there are functions B which satisfy (1. 4), (1. 5), and (1. 6).

1. 4. If J has exactly one element (the case of a single contraction) then the values of B are completely determined by (1. 4) and (1. 5). So in this case there is a solution (unique) if and only if the positivity condition holds for these values of B. Sz.-Nagy proved that positivity does hold in this case, solving the problem completely for a single contraction.

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[6]) The left side of (1. 6) is equal to $\sum_{i=1}^{N} \|U(n_i)x_i\|^2$.

[7]) After the U_α have been constructed, (n, x) will turn out to be the value of $U(n)x$.

[8]) This discussion is valid whether the scalars are the real, complex or quaternionic numbers.

1. 5. If J has more than one element it is not yet known whether solutions of (1. 1), (1. 2) always exist, even for the case of commuting T_α. However, SZ.-NAGY extended his methods to the case of doubly commuting T_α (that is: $T_\alpha T_\beta^* = T_\beta^* T_\alpha$ as well as $T_\alpha T_\beta = T_\beta T_\alpha$ for $\alpha \neq \beta$). In order to describe SZ.-NAGY's result more precisely, let us call U_α $(\alpha \in J)$ a *Sz.-Nagy—Brehmer* (abbreviation: *Sz.-N.—B.*) *dilation of* T_α $(\alpha \in J)$ if (stronger than (1. 1)):

$$(1.7)\qquad \begin{cases} (T(m))^* T(n)x = P_H(U(m))^* U(n)x \\ \textit{for all } x \in H \textit{ and all positive-disjoint } n, m. \end{cases}$$

Clearly if the U_α $(\alpha \in J)$ are a Sz.-N.—B. dilation then the function B is completely determined by the T_α $(\alpha \in J)$.

Now for given T_α $(\alpha \in J)$ (without assumption as to existence of a unitary dilation) let us call $(T(n)x \mid T(m)y)$ the *Sz.-Nagy—Brehmer* (abbreviation: Sz.-N.—B.) *values* for $B(n, x; my)$. Then, if T_α $(\alpha \in J)$ does possess a Sz.-N.—B. minimal dilation, this dilation is obviously determined (uniquely) by the Sz.-N.—B. values. On the other hand, the Sz.-N.—B. values always satisfy (1. 4) and (1. 5); so if they determine a unitary dilation at all (that is, if the positivity condition (1. 6) holds) then this dilation is necessarily a Sz.-N.—B. dilation.

SZ.-NAGY proved: with the Sz.-N.—B. values, positivity does hold *if the* T_α $(\alpha \in J)$ *are doubly commuting:* thus for this case he obtained a particular solution of (1. 1), (1. 2), in fact a Sz.-N.—B. dilation. We note: if U_α $(\alpha \in J)$ is a Sz.-N.—B. dilation then the *stronger* condition

$$(1.7)'\qquad \begin{cases} (T(n_1))^* T(n_2)(T(n_3))^* \dots T(n_s)x = P_H(U(n_1))^* U(n_2)(U(n_3))^* \dots U(n_s)x \\ \textit{for all } x \in H \textit{ and pairwise positive-disjoint } n_1, \dots, n_s \end{cases}$$

holds if and only if the T_α $(\alpha \in J)$ are doubly commuting.

1. 6. Recently BREHMER [1] has refined SZ.-NAGY's method and has shown that the Sz.-N.—B. values satisfy the positivity condition (and hence yield a solution for (1. 7), (1. 2) but not for (1. 7)′ in general), assuming commutativity of the T_α $(\alpha \in J)$ together with a condition weaker than that of double commutativity. SZ.-NAGY and BREHMER use a set of indices $J = \{\alpha\}$ not assumed to be ordered; but in fact, when the T_α are commuting it is equivalent to consider J as totally ordered, in any way.

1. 7. In the present paper we give a new and simple proof of positivity for Sz.-N.—B. values. More precisely, we show:

If the T_α $(\alpha \in J)$ *are commuting operators then positivity holds for the Sz.-Nagy—Brehmer values if and only if the following condition holds: For every finite subset of J, denoted* $1, \dots, r$ *for convenience, the operator*

$$(1.8)\qquad \begin{cases} P(T_1, \dots, T_r) \equiv \sum\limits_u (-1)^{a(u)} (T(u))^* T(u) \textit{ shall be positive definite,} \\ \textit{where } u = (u_1, \dots, u_r) \textit{ (varies over all } 0 \leqq u_s \leqq 1,\ s = 1, \dots, r), \textit{ and } a(u) = \\ = \sum\limits_{s=1}^{r} u_s.\,^{9)} \end{cases}$$

[9] If $r = 1$, (1. 8) becomes $P(T) \geqq 0$, that is $1 - T^*T \geqq 0$, which is equivalent to the assertion: T is a contraction.

Our proof (see section 2, 3 below) depends on some simple matrix calculations and is advantageous even for the case of a single contraction (where the condition (1. 8) is satisfied trivially). Like SZ.-NAGY and BREHMER, we do not use spectral theory or square roots of positive definite operators. But to show that (1. 8) is satisfied by doubly commuting contractions seems to require (as in the proof of SZ.-NAGY, BREHMER) the fact that the product of two commuting positive definite operators is positive definite. An elementary proof of this fact was given by F. RIESZ [6].

1. 8. If J contains only two elements, BREHMER's condition coincides with our (1. 8); if J contains more than two elements, BREHMER's condition, (10) of his paper [1], is our (1. 8) but for $kT_1, \ldots, kT_r$ in place of $T_1, \ldots, T_r$ for all $0<k<1$ (and hence, by continuity also for $k=1$, which is precisely our (1. 8)). It is not clear whether BREHMER's (apparently) stronger condition is actually necessary when J contains more than two elements.

The proof of BREHMER (like that of SZ.-NAGY) employs Fourier series and seems to be valid only for complex (or real) scalars. Our proof is valid for quaternionic scalars also.

1. 9. In section 4 we discuss possible properties of a particular T_β (to be isometric or to double commute with the other T_α $(\alpha\neq\beta)$) which permit omitting this T_β from the condition (1. 8). In section 5 we give some illustrative examples including cases of T_α $(\alpha\in J)$ which are *not* commuting but do possess a Sz.-N.—B. dilation (the order of J is essential in these cases.)

2. Proof of positivity for a single contraction T

2. 1. With the notation of section 1. 2 we need only verify (1. 6). It is easy to see that for a single contraction, a statement equivalent to (1. 6) is this: For each integer $N\geqq 1$, and arbitrary $x_0, \ldots, x_N$ in H

$$\sum_{\substack{i,j=0\\ i>j}}^{N} (T^{i-j}x_i|x_j) + \sum_{\substack{i,j=0\\ j\geqq i}}^{N} (x_i|T^{j-i}x_j) \geqq 0. \tag{2. 1}$$

2. 2. Let $\bar{H}$ be the direct sum of $N+1$ copies of H, so that the vectors in $\bar{H}$ can be identified with the systems $(x_0, \ldots, x_N)$ (with all x_i in H). Then (2. 1) clearly asserts that a certain bounded linear operator $\bar{T}$ on $\bar{H}$ is positive definite, namely the $\bar{T}$ whose matrix[10]) has (i,j)-th entry

$$\bar{T}_{i,j}=T^{j-i} \quad \text{if} \quad j\geqq i, \qquad =(T^*)^{i-j} \quad \text{if} \quad j<i.$$

2. 3. We shall show that $\bar{T}=W^*DW$ for some W and some positive definite D. This will imply that $\bar{T}$ is positive definite.[11]) For this purpose we choose W, D

[10]) We use the same symbol to denote the operator and the representing matrix of the operator.

[11]) The fact that this matrix $\bar{T}$ is positive definite for arbitrary contraction T seems not to be mentioned in the literature even for the case that H is one-dimensional (then $\bar{T}$ is a numerical matrix and T is simply an arbitrary scalar of absolute value $\leqq 1$).

as follows:

$$\begin{aligned} W_{i,j} &= T^{j-i} && \text{if } j \geqq i, \\ &= 0 && \text{if } j < i; \\ D_{i,j} &= 1 && \text{if } i = j = 0, \\ &= 1 - T^*T && \text{if } i = j > 0, \\ &= 0 && \text{if } i \neq j. \end{aligned}$$

Since D is a diagonal matrix with $1, 1-T^*T, \ldots, 1-T^*T$ on the diagonal and since $1-T^*T$ is positive definite on H (T is a contraction), therefore D is positive definite on $\bar{H}$.

To confirm that $\bar{T} = W^*DW$ we note that $\bar{T}$ and W^*DW are Hermitian symmetric and so we need only prove that their (i,j)-th entries coincide for $i \leqq j$. Now, for $i \leqq j$,[12])

$$(W^*DW)_{i,j} = \sum_{h=0}^{i} (W^*)_{i,h} D_{h,h} W_{h,j} = (T^*)^i T^j + \sum_{h=1}^{i} (T^*)^{i-h}(1-T^*T)T^{j-h} =$$

$$= (T^*)^i T^j + \sum_{h=1}^{i} \left((T^*)^{i-h} T^{j-h} - (T^*)^{i-h+1} T^{j-h+1}\right) =$$

$$= (T^*)^0 T^{j-i} = T^{j-i} = \bar{T}_{i,j}.$$

Thus $\bar{T} = W^*DW$ as stated and so positivity is established for a single contraction.

3. More than one contraction

3.1. The Sz.-N.—B. values determine the solution (unique if existing) of (1.2) and

$$(3.1) \qquad (U(n)x \mid U(m)y) = (T(n)x \mid T(m)y) \; \textit{for positive-disjoint } n, m.$$

Condition (3.1) is equivalent to (1.7) and stronger than (1.1).

3.2. As we have seen above, such a (Sz.-N.—B.) solution does exist if and only if positivity holds and this condition can be expressed in the following way.

Let r, N be arbitrary integers $\geqq 1$[13]) and let $1, \ldots, r$ denote any finite subordered subset of J (we do not yet require the T_α to be commuting).

Let i denote an r-tuple: $i = (i_1, \ldots, i_r)$ with $0 \leqq i_s \leqq N$ for each s, and for each such i let x_i be an arbitrary vector in H.

Then the positivity condition (1.6) is equivalent to the statement: For all such $\{x_i\}$:

$$(3.2) \qquad \sum_{i,j} (T_1^{p_1} \ldots T_r^{p_r} x_i \mid T_1^{q_1} \ldots T_r^{q_r} x_j) \geqq 0$$

where $p_s \equiv p_s(i,j) = \max(0, i_s - j_s)$ and $q_s(i,j) = p_s(j,i)$.

[12]) Note: $(W^*)_{i,h} = (W_{h,i})^* = 0$ if $h > i$.

[13]) If $r = 1$ the following discussion will specialize to that of section 2.

3. 3. Let $\bar{H} \equiv \bar{H}_r$ be the direct sum of $(N+1)^r$ copies of H so that the elements of $\bar{H}$ can be identified with the systems $\{x_i;\, i \text{ varying}\}$. As in section 2, (3. 2) asserts that a certain operator $\bar{T}$ on $\bar{H}$ is positive definite, namely the $\bar{T}$ whose $(N+1)^r \times \times (N+1)^r$ matrix has (i, j)-th entry:

$$\bar{T}_{i,j} = (T_r^*)^{p_r} \ldots (T_1^*)^{p_1} T_1^{q_1} \ldots T_r^{q_r}$$

where $p_s = p_s(i, j)$, $q_s = q_s(i, j)$ are as defined in section (3. 2) (note: for each s at least one of p_s, q_s must be 0).

3. 4. We shall show that if the T_α $(\alpha \in J)$ are commuting operators, then (i): $\bar{T} = W^* D W$ holds always (without additional restrictions on the T_α) where D is a certain diagonal operator, and (ii): $\bar{T}$ is positive definite if and only if this D is positive definite.

For this purpose we define W, D by:

$$\begin{aligned} W_{i,j} &= T_1^{j_1 - i_1} \ldots T_r^{j_r - i_r} \text{ if for each } s = 1, \ldots, r,\ i_s \leqq j_s, \\ &= 0 \text{ otherwise;} \\ D_{i,j} &= 0 \qquad\qquad\qquad \text{if } i \neq j, \\ &= P(T_{s_1}; \ldots, T_{s_t}) \text{ if } i = j \text{ and the } s \text{ with } i_s > 0 \text{ are } s_1, \ldots, s_t. \end{aligned}$$

We recall that $P(T_1, \ldots, T_r)$ was defined in (1. 8) of section 1. From the definition it follows immediately that if $r > 1$ then

$$P(T_1, \ldots, T_r) = P(T_1, \ldots, T_{r-1}) - T_r^* P(T_1, \ldots, T_{r-1}) T_r.$$

3. 5. We shall show that $\bar{T} = W^* D W$ by induction on r. For $r = 1$ the equality was proved in section 2 above. Suppose now that $r > 1$ and that the equality has been established for $r - 1$ in place of r.

Our present $\bar{H} \equiv \bar{H}_r$ may be considered as the direct sum of $N+1$ copies of $\bar{H}_{r-1}$, the different copies being associated with the possible values of $i_r = 0, 1, \ldots, N$. Each of $\bar{T}$, W^*, D, W can be represented by an $(N+1) \times (N+1)$ matrix with (s, t)-th entry $(s, t = 0, \ldots, N)$ an operator on $\bar{H}_{r-1}$.

If we use the indices r, $r-1$ to refer (in the obvious way) to the situation for $T_1, \ldots, T_r, \bar{H}_r$ and $T_1, \ldots, T_{r-1}, \bar{H}_{r-1}$ respectively, we have:

$$\begin{aligned} (\bar{T}_r)_{s,t} &= (T_r^*)^{s-t} \bar{T}_{r-1} && \text{if } s \geqq t, \\ &= \bar{T}_{r-1} T_r^{t-s} && \text{if } s < t; \\ (W_r)_{s,t} &= W_{r-1} T_r^{t-s} && \text{if } s \leqq t, \\ &= 0 && \text{if } s > t; \\ (D_r)_{s,t} &= D_{r-1} && \text{if } s = t = 0, \\ &= D_{r-1} - T_r^* D_{r-1} T_r && \text{if } s = t > 0, \\ &= 0 && \text{if } s \neq t. \end{aligned}$$

Since $\bar{T} \equiv \bar{T}_r$ and $W^*DW \equiv W_r^* D_r W_r$ are clearly Hermitian symmetric (D_r is diagonal with Hermitian symmetric diagonal elements), so we will know that $\bar{T}_r = W_r^* D_r W_r$ if their (s, t)-th entries coincide for $s \leqq t$. Now, for $s \leqq t$,

$$(W_r^* D_r W_r)_{s,t} = \sum_{h=0}^{s} (W_r^*)_{s,h} (D_r)_{h,h} (W_r)_{h,t} = {}^{14)}$$

$$= (W_r^*)_{s,0} D_{r-1} (W_r)_{0,t} + \sum_{h=1}^{s} (W_r^*)_{s,h} (D_{r-1} - T_r^* D_{r-1} T_r)(W_r)_{h,t} =$$

$$= (T_r^*)^s W_{r-1}^* D_{r-1} W_{r-1} T_r^t + \sum_{h=1}^{s} \left((T_r^*)^{s-h} W_{r-1}^* D_{r-1} W_{r-1} T_r^{t-h} - \right.$$

$$\left. - (T_r^*)^{s-h} W_{r-1}^* T_r^* D_{r-1} T_r W_{r-1} T_r^{t-h}\right).$$

Now T_r commutes with W_{r-1} since the entries of W_{r-1} are products of $T_1, \ldots, T_{r-1}$ only.[15] Also $W_{r-1}^* D_{r-1} W_{r-1} = \bar{T}_{r-1}$ by our inductive assumption. Hence

$$(W_r^* D_r W_r)_{s,t} = (T_r^*)^s \bar{T}_{r-1} T_r^t + \sum_{h=1}^{s} \left((T_r^*)^{s-h} \bar{T}_{r-1} T_r^{t-h} - (T_r^*)^{s-h+1} \bar{T}_{r-1} T_r^{t-h+1}\right) =$$

$$= (T_r^*)^0 \bar{T}_{r-1} T_r^{t-s} = (\bar{T}_r)_{s,t}.$$

This completes the proof that $\bar{T} = W^*DW$.

3. 6. Next we show that for $\bar{T}$ to be positive definite it is necessary and sufficient that D be positive definite. Since $\bar{T} = W^*DW$ the "sufficiency" is obvious.

The "necessity" would follow at once if W had a right inverse W^{-1}: $WW^{-1} = 1$. For then, for every x in $\bar{H}$,

$$(Dx|x) = (DWW^{-1}x | WW^{-1}x) = \left(W^*DW(W^{-1}x) | (W^{-1}x)\right) \geqq 0.$$

Now we shall show that W can be considered as a semi-diagonal matrix with 1 at each diagonal place. It is easy to see that every such W has an inverse.

To exhibit W as a semi-diagonal matrix we recall that the entries $W_{i,j}$ are indexed by r-tuples i, j. We totally order these r-tuples by the relation $(i_1, \ldots, i_r) \ll (j_1, \ldots, j_r)$ if $i_1 = j_1, \ldots, i_{s-1} = j_{s-1}, i_s < j_s$ for some $s = 1, \ldots, r$.

With this total ordering of the indices, W is upper semi-diagonal, that is $W_{i,j} = 0$ if $j \ll i$ (in fact, $W_{i,j} = 0$ if $j_s < i_s$ for some s). Also, the diagonal elements of W are $W_{i,i}$, all $= 1$.

This completes the proof that $\bar{T}$ is positive definite if and only if D is positive definite.

3. 7. Since D is diagonal it is positive definite if and only if each of its diagonal entries is positive definite. These entries are all of the form $P(T_1, \ldots, T_s)$ for some

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14) Note: $(W_r^*)_{s,h} = (W_r)_{h,s}^* = 0$ if $h > s$.

15) It is at this point in our argument that we need the hypothesis that the $T_\alpha (\alpha \in J)$ are commuting.

$T_1, \ldots, T_s$ and clearly every $P(T_1, \ldots, T_r)$ does occur as a diagonal element in some D. So we have our final result:

Theorem. *For given commuting operators T_α ($\alpha \in J$) in order that the Sz.-Nagy—Brehmer values determine a unitary dilation it is necessary and sufficient that*

$$P(T_1, \ldots, T_r) \geqq 0 \tag{1.8}$$

for all finite subsets $1, \ldots, r$ *of* J.

4. Some comments on the condition (1. 8)

4. 1. Let J_1 be the subset of J remaining after discarding all α for which T_α is isometric, i. e. for which $T_\alpha^* T_\alpha = 1$. Then (1. 8) for J is implied by (1. 8) for J_1.[16]

Indeed, if $r > 1$ and T_1 is isometric then

$$P(T_1, \ldots, T_r) =$$
$$= \sum \pm \left((T_r^*)^{u_r} \ldots (T_2^*)^{u_2} T_1^* T_1 T_2^{u_2} \ldots T_r^{u_r} - (T_r^*)^{u_r} \ldots (T_2^*)^{u_2} T_2^{u_2} \ldots T_r^{u_r} \right) = \sum \pm (0) = 0.$$

More generally, if T_α ($\alpha \in J$) are arbitrary contractions (not required to be commuting) and J is ordered then the existence of a Sz.-N.—B. dilation is not affected by discarding the commuting isometries (the T_α with properties: $T_\alpha^* T_\alpha = 1$ and $T_\alpha T_\beta = T_\beta T_\alpha$ for all $\beta \in J$). Moreover this fact could have been verified at the *beginning* of our discussion without use of the general condition (1. 8).

To see this consider the condition (3. 2) and suppose that T_1 is a commuting contraction. Then (3. 2) can be expressed:

$$\sum_{\substack{i_2, \ldots, i_r \\ j_2, \ldots, j_r}} \Big(\sum_{i_1 \geqq j_1} (T_2^{p_2} \ldots T_r^{p_r} T_1^{i_1 - j_1} x_{i_1, \ldots, i_r} | T_2^{q_2} \ldots T_r^{q_r} x_{j_1, \ldots, j_r}) +$$
$$+ \sum_{i_1 < j_1} (T_2^{p_2} \ldots T_r^{p_r} x_{i_1, \ldots, i_r} | T_2^{q_2} \ldots T_r^{q_r} T_1^{j_1 - i_1} x_{j_1, \ldots, j_r}) \Big) \geqq 0.$$

Since $T_1^{i_1 - j_1} = (T_1^*)^{j_1} T_1^{i_1}$ if $i_1 > j_1$, this condition can be expressed as

$$\sum_{\substack{i_2, \ldots, i_r \\ j_2, \ldots, j_r}} (T_2^{p_2} \ldots T_r^{p_r} x_{i_2, \ldots, i_r} | T_2^{q_2} \ldots T_r^{q_r} x_{j_2, \ldots, j_r}) \geqq 0$$

where $x_{i_2, \ldots, i_r}$ denotes $\sum_{i_1} T_1^{i_1} x_{i_1, \ldots, i_r}$.

This proves our statement.

4. 2. Let J_2 be the subset of J_1 remaining after discarding from J_1 all α for which T_α double commutes with all other T_β ($\beta \in J_1$, $\beta \neq \alpha$). Then (1. 8) for J is implied by (1. 8) for J_2.

For if T_1 double commutes with $T_2, \ldots, T_r$ then T_1 commutes with $P(T_2, \ldots, T_r)$ and hence

$$\begin{aligned} P(T_1, \ldots, T_r) &= P(T_2, \ldots, T_r) - T_1^* T_1 P(T_2, \ldots, T_r) \\ &= (1 - T_1^* T_1) P(T_2, \ldots, T_r). \end{aligned}$$

[16]) In terms of BREHMER's condition this was pointed out by SZ.-NAGY [5].

Now $1-T_1^*T_1$ is positive definite and commutes with $P(T_2, \dots, T_r)$. Hence $P(T_1, \dots, T_r)$ is positive definite if $P(T_2, \dots, T_r)$ is positive definite; here we use the fact that the product of two commuting positive definite operators is positive definite (see [6]).

More generally, if T_α $(\alpha \in J)$ are arbitrary contractions and J is ordered then the existence of a Sz.-N. — B. dilation is not affected by discarding the doubly commuting members (the T_α with properties: $T_\alpha T_\beta = T_\beta T_\alpha$ and $T_\alpha T_\beta^* = T_\beta^* T_\alpha$ for all $\beta \neq \alpha$). Moreover this fact could have been verified at the end of section 2 (discussion for a single contraction), without use of the general condition (1. 8).

To see this, consider the condition (3. 2) and suppose that T_1 is doubly commuting. Then (3. 2) can be expressed as follows (use the notation $u=i_1$, $v=j_1$, $s=(i_2, \dots, i_r)$, $t=(j_2, \dots, j_r)$, $L_{u,v}=T_1(i_1-j_1)$, $M_{s,t}=(T_2^{q_2} \dots T_r^{q_r})^* T_2^{p_2} \dots T_r^{p_r}$):

$$\sum_{u,s,v,t} (M_{s,t} L_{u,v} x_{u,s} | x_{v,t}) \geqq 0.$$

Let L be the linear operator in $\sum_u \oplus H_u$ (with each H_u a copy of H) such that for all fixed s, t:

$$(L(x_{u,s};\ u \text{ varying}) | (y_{v,t};\ v \text{ varying})) = \sum_{u,v} (L_{u,v} x_{u,s} | y_{v,t}).$$

Similarly let M be the linear operator in $\sum_s \oplus H_s$ (each H_s a copy of H) such that for fixed u, v:

$$(M(x_{u,s};\ s \text{ varying}) | (y_{v,t};\ t \text{ varying})) = \sum_{s,t} (M_{s,t} x_{u,s} | y_{v,t}).$$

Then L, M determine, in the obvious way, commuting Hermitian symmetric operators in $\sum_{u,s} \oplus H_{u,s}$ (with each $H_{u,s}$ a copy of H) and for all x, y in $\sum_{u,s} \oplus H_{u,s}$:

$$(LMx | y) = \sum_{u,v,s,t} (M_{s,t} L_{u,v} x_{u,s} | y_{v,t}).$$

Since L is positive definite (by section 2), it follows that LM is positive definite if M is positive definite. This proves our statement.

4. 3. The condition (1. 8) for J_2 (and hence for J) will hold if

$$\sum_{\alpha \in J_2} \|T_\alpha\|^2 \leqq 1.\text{[17]}$$

For if $0 \leqq P(T_1, \dots, T_r) \leqq 1$ then $(P(T_1, \dots, T_r) y | y) \leqq (y | y)$ for all y; $T_{r+1}^* P(T_1, \dots, T_r) T_{r+1} \leqq T_{r+1}^* T_{r+1}$; and so

(i) $$P(T_1, \dots, T_{r+1}) = P(T_1, \dots, T_r) - T_{r+1}^* P(T_1, \dots, T_r) T_{r+1} \geqq$$
$$\geqq P(T_1, \dots, T_r) - T_{r+1}^* T_{r+1}$$

(ii) $$P(T_1, \dots, T_{r+1}) \leqq P(T_1, \dots, T_r) \leqq 1.$$

[17]) In terms of BREHMER's condition this was shown by BREHMER [1] (see also SZ.-NAGY [5]).

Now suppose $\sum_i T_i^* T_i \leqq 1$. Then $0 \leqq 1 - T_1^* T_1 = P(T_1) \leqq 1$ and by induction on r:

$$0 \leqq 1 - \sum_{i=1}^{r} T_i^* T_i \leqq P(T_1, \ldots, T_r) \leqq 1$$

for all $r \geqq 1$. Hence $P(T_1, \ldots, T_r) \geqq 0$ for all $T_1, \ldots, T_r$.

5. Some examples

5. 1. If K, U_α $(\alpha \in J)$ is a solution for (1. 1), (1. 2) for T_α $(\alpha \in J)$ then K, U_α^* $(\alpha \in J)$ is a solution for T_α^* $(\alpha \in J^*)$ where J^* is identical with J except that the order is inverted; and if $T_\alpha T_\beta^* = T_\beta^* T_\alpha$ for $\alpha \neq \beta$ then if one of these solutions is a Sz.-N. — B. solution so is the other one.

Since T_α $(\alpha \in J)$ are commuting if and only if T_α^* $(\alpha \in J^*)$ are commuting, it follows that (1. 1), (1. 2) have a particular solution which could be called a *dual Sz.-Nagy—Brehmer dilation if the T_α are commuting and satisfy condition* (1. 8)* (this means: (1. 8) for the T_α^*).

If the T_α $(\alpha \in J)$ are commuting and satisfy both (1. 8) and (1. 8)* we obtain thus two particular solutions of (1. 1), (1. 2) for the T_α and it is easy to see that these will coincide if and only if the T_α are doubly commuting.

Consider the special example: $J = \{1, 2\}$, $T_1 = T_2 = T$, where T is the operator on the two dimensional space spanned by basis φ_1, φ_2 with $T\varphi_1 = k\varphi_2$, $T\varphi_2 = 0$ with $k^2 \leqq \frac{1}{2}$. In this example T_1 and T_2 commute and both (1. 8) and (1. 8)* hold, but $T_1 T_2^* \neq T_2^* T_1$. The Sz.-N. — B. solution for T_2^*, T_1^* yields a dilation for T_1, T_2 which is dual Sz.-N. — B. but *not* Sz.-N. — B. for T_1, T_2.

In the preceding example, if $k^2 > \frac{1}{2}$ then T_1 and T_2 commute but neither (1. 8) nor (1. 8)* are satisfied; in this example (1. 1), (1. 2) do have a solution namely K, U_1, U_2 where K, U is the solution for T (single contraction) and $U_1 = U_2 = U$, but of course this solution is neither a Sz.-N. — B. nor a dual Sz.-N. — B. dilation for T_1, T_2.

5. 2. Suppose U_α $(\alpha \in J$, J totally ordered) is a Sz.-N. — B. minimal dilation for T_α $(\alpha \in J)$. Suppose J_1, J_2 are complementary subsets of J such that $\alpha_1 < \alpha_2$ for all $\alpha_1 \in J_1, \alpha_2 \in J_2$ (we permit J_1 or J_2 to be empty), and let $S_\alpha = T_\alpha^*$, $V_\alpha = U_\alpha^*$ if $\alpha \in J_1$ and let $S_\alpha = T_\alpha$, $V_\alpha = U_\alpha$ if $\alpha \in J_2$.

Let J_{12} coincide with J but with order as follows: if $\alpha \in J_1$ then $\alpha \leqq \alpha_1$ in J shall imply $\alpha_1 \leqq \alpha$ in J_{12} and if $\alpha_2 \in J_2$ then $\alpha \leqq \alpha_2$ in J shall imply $\alpha \leqq \alpha_2$ in J_{12}.

Then V_α $(\alpha \in J_{12})$ will be a Sz.-N. — B. minimal dilation for S_α $(\alpha \in J_{12})$ provided $T_\alpha^* T_\beta = T_\beta T_\alpha^*$ for all $\alpha \in J_1, \beta \in J_1, \alpha \neq \beta$ (if J_1 is empty or consists of one element this condition is vacuous).

Hence if T_α $(\alpha \in J)$ are *commuting* contractions satisfying the condition (1. 8) and if the T_α $(\alpha \in J_1)$ are doubly commuting then the S_α $(\alpha \in J_{12})$ possess a Sz.-N. — B. dilation. However the S_α $(\alpha \in J_{12})$ need not be commuting.

For example, consider two contractions T_1, T_2 such that T_1^*, T_2 commute but T_1, T_2 do not commute. If $1 - T_1 T_1^* - T_2^* T_2 + T_2^* T_1 T_1^* T_2 \geqq 0$ then T_1^*, T_2 possess a Sz.-N. — B. dilation, V_1, V_2 say, and then V_1^*, V_2 will be a Sz.-N. — B. dilation for the original not-commuting T_1, T_2.

In a subsequent paper to appear in *Duke Journal of Mathematics* we shall investigate conditions under which a (wider) class of not-commuting contractions possesses a (mixed Sz.-N. — B.) dilation.

Added in proof: A paper by the writer entitsld: "Intrinsie description of the Sz.- Nagy—Brehmer unitarg dilation", to appear in *Studia Mathematica*, gives a different proof of the Theorem given above (end of section 3), and shows the geometrical significance of the condition (1.8).

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[6] F. RIESZ, Über die linearen Transformationen des complexen Hilbertschen Raumes, *Acta Sci. Math.*, **5** (1930—32), 23—54 (see footnote 9 there; it is valid if the scalars are real, complex or quaternionic).

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